Borromean Entanglement Revisited

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Abstract

An interesting analogy between quantum entangled states and topological links was suggested by Aravind. In particular, he emphasized a connection between the Greenberger-Horne-Zeilinger (GHZ) state and the Borromean rings. However, he made the connection in a way that depends on the choice of measurement basis. We reconsider it in a basis-independent way by using the reduced density matrix.

1 Introduction

Quantum entanglement is a form of correlation among quantum systems which cannot be described by any classical (local reality) theory. This strange nonlocality in quantum mechanics, which Albert Einstein called ”spooky action at a distance”, is now considered to be the key resource for quantum information processing such as quantum computation and quantum cryptography.

It seems quite natural to think of analogies between quantum entanglement and topological entanglement, and some authors actually suggested such analogies [1, 2]. Among them, we pick up the idea of Aravind [1] in this paper. He made the correspondence between quantum states and topological links by associating each particle with a ring, and measurement of a particle with cutting of the corresponding ring. However, there are many possible measurements of a particle, and the correspondence depends on the choice of the measurement. We reconsider it by using the partial trace of the density matrix instead of the measurement.

In the following, we first review the basics of quantum mechanics for convenience of the readers who are not familiar with quantum mechanics and entanglement. Then we proceed to the discussion on the correspondence between quantum states and links.

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2 Basic concepts and tools

In this section we introduce basic concepts and tools of quantum mechanics according to Chapter 3 of the text book by Nielsen and Chuang [3].

2.1 Postulates of quantum mechanics

The mathematical structure of quantum mechanics can be summarized in the following four postulates.

Postulate 1 *Associated to any isolated physical system is a complex vector space with inner product (that is a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.*

We use the standard quantum mechanical notation $|\psi\rangle$ to represent a vector, where ψ is a label for the vector. Its dual vector is denoted as $\langle\psi|$. $\langle\phi|\psi\rangle$ represents the inner product between the vectors $|\phi\rangle$ and $|\psi\rangle$. Note that the product of the form $|\phi\rangle\langle\psi|$ is regarded as an operator which acts on a vector $|v\rangle$ as $|\phi\rangle\langle\psi|v\rangle$.

Postulate 2 *The evolution of a closed quantum system is described by a unitary transformation.*

Postulate 3 *Ideal quantum measurements are described by a collection $\{P_m\}$ of projection operators acting on the state space of the system. The index m refers to the measurement outcomes. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is*

$$p(m) = \langle\psi|P_m|\psi\rangle, \tag{1}$$

and the state of the system after the measurement is

$$\frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}. \tag{2}$$

The projection operators satisfy the relation

$$\sum_m P_m = I, \tag{3}$$

where I is the identity operator, so that the probabilities sum to one.

Postulates 3 describes only ideal (projective) measurements, but actual measurements are rarely ideal. Therefore some text books including [3] adopt more general description of measurements from the beginning. However, general measurements can be represented by combining projective measurements and other postulates of quantum mechanics. Since we use only projective measurements in this paper, we do not include general measurements in Postulate 3 for simplicity.

Postulate 4 *The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered* 1 *through* n*, and system number* i *is prepared in the state* $|\psi_i\rangle$*, then the joint state of the total system is* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$*.*

We often use abbreviated notations $|\psi_1\rangle|\psi_2\rangle$ or $|\psi_1\psi_2\rangle$ to denote $|\psi_1\rangle \otimes |\psi_2\rangle$.

In the following, we consider composite systems consisting of *qubits*. A qubit (quantum bit) is the simplest quantum mechanical system whose state space is two-dimensional. It is used as a unit of quantum information. We denote the basis vectors of the two-dimensional Hilbert space as $|0\rangle$ and $|1\rangle$.

2.2 The density operator

The state vector formalism introduced in the last subsection can be used to describe a system whose state is completely specified. In such a case, the state is called a *pure state*. In most situations, however, we do not have complete knowledge about the state. The density operator can be used to describe such a situation.

Suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where i is an index, with respective probabilities p_i. The set $\{p_i, |\psi_i\rangle\}$ is called an ensemble of pure states. The density operator for the system is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{4}$$

If $p_i = 1$ for an index i and all other probabilities are zero, the density matrix

$$\rho = |\psi_i\rangle\langle\psi_i| \tag{5}$$

describes a pure state corresponding to the state vector $|\psi_i\rangle$. Otherwise the density operator ρ represents a *mixed state*.

In the density operator formalism, time evolution of a quantum system is described by a unitary operator U as $\rho \to U\rho U^\dagger$. The probability that the result labeled by m occurs is $\mathrm{tr}(\rho P_m)$, and the the state of the system after the measurement is $\frac{P_m \rho P_m}{\sqrt{\mathrm{tr}(\rho P_m)}}$. It is easy to check that the density matrix formalism reduces to the state vector formalism when it represents a pure state.

An operator can be regarded as a density operator if and only if it is a positive operator with $\mathrm{tr}\rho = 1$. If an operator satisfies this condition, it is obvious that the operator can be written in the form (4), where p_i is an eigenvalues of ρ and $|\psi_i\rangle$ is the corresponding eigenvector. Therefore ρ is the density operator associated to the ensemble $\{p_i, |\psi_i\rangle\}$. Note, however, that the correspondence between density operators and ensembles is not one-to-one. There are infinitely many ensembles corresponding to a density operator.

2.3 The reduced density operator

When we observe only a subsystem of a composite system, the subsystem is described by the reduced density operator. Suppose we have a physical system composed of subsystems A and B, and its state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A \equiv \mathrm{tr}_B \left(\rho^{AB}\right), \tag{6}$$

where tr_B is the *partial trace* over system B. The partial trace is defined by

$$\mathrm{tr}_B \left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) \equiv |a_1\rangle\langle a_2| \, \mathrm{tr}\left(|b_1\rangle\langle b_2|\right), \tag{7}$$

where $|a_1\rangle$ and $|a_2\rangle$ ($|b_1\rangle$ and $|b_2\rangle$) are any two vectors in the state space of A (B). In addition, we require the linearity of the partial trace, which completes the specification.

2.4 Entanglement

Let us consider a pure state of a system composed of subsystems A and B. If the state vector of the system can be written as a tensor product

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle, \tag{8}$$

where $|\psi_A\rangle$ ($|\psi_B\rangle$) is some state vector of subsystem A (B), the state is called *separable*. It is known that if a pure state is not separable, there exists a Bell-type inequality which is violated by this state [4]. It means that the state has a correlation which cannot be described by any classical (local reality) theory. Such a state is called *entangled*. For example, it is not difficult to show that the following 2-qubit state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \ + \ |1\rangle \otimes |1\rangle) \tag{9}$$

cannot be written as a tensor product of two single-qubit states. Hence this state is entangled.

An mixed state is called separable if it can be decomposed into the following form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \tag{10}$$

where ρ_i^A (ρ_i^B) is a density matrix of subsystem A (B), $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. Since $\rho_i^A$and ρ_i^B can be represented by ensembles of pure states, (10) means that a separable mixed state can be represented by an ensemble of separable pure states.

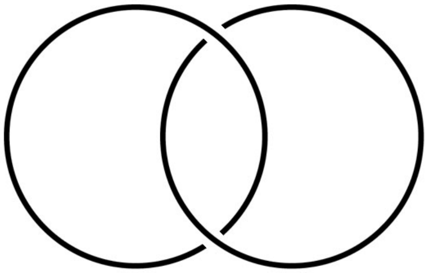

Figure 1: Entangled two rings corresponding to entangled two qubits.

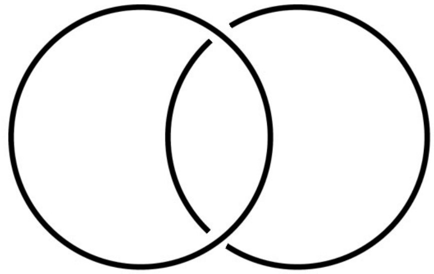

Figure 2: Unentangled two rings corresponding to unentangled two qubits.

In this paper, a non-separable mixed state is called entangled. Note, however, that the definition of entanglement is not so trivial for mixed states. For example, a separable mixed state satisfies all Bell-type inequalities, but the converse does not hold [5].

In general, it is a difficult task to determine if a given mixed state is separable or not. However, there are some simple criteria for 2-qubit systems. One is Peres' criterion based on the positivity of the partial transposed density matrix [6]. Another useful criterion is based on the *concurrence* [7], which is given by

$$C(\rho) = \max\{0, \ \ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{11}$$

Here, λ_is are the square roots of the eigenvalues of $\tilde{\rho}\rho$ in decreasing order, where

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{12}$$

and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{13}$$

A state represented by the density operator ρ is separable if and only if $C(\rho) = 0$.

3 Quantum and topological entanglement

Let us introduce a correspondence between quantum states and and links. In the following, we consider composite systems consisting of qubits, and associate a ring with a qubit. Entangled two qubits are represented by entangled two rings (Fig. 1), and separable two qubits are represented by unentangled two rings (Fig. 2).

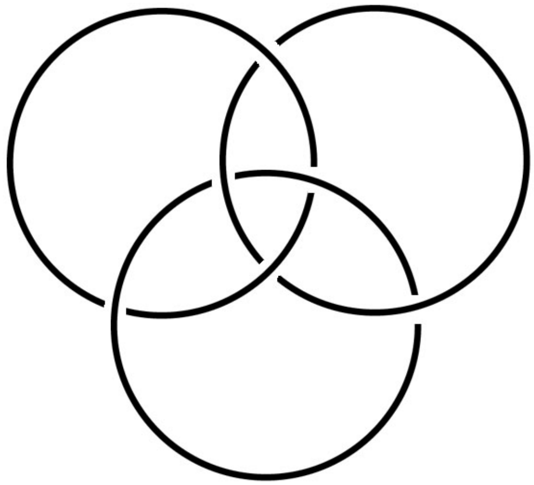

Figure 3: The Borromean rings. If one of the three rings is cut, the other two rings can be pulled apart.

Then let us consider 3-qubit systems. The three qubits are named A, B and C, and the tensor product of the states of three qubits $|\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$ is denoted as $|\psi_A\psi_B\psi_C\rangle$, where ψ_A, ψ_B and ψ_C are labels for the states of A, B and C, respectively.

As the first example, we consider the Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{14}$$

This state is entangled, hence the corresponding three rings are also tangled somehow.

Aravind considered a measurement of a qubit whose measurement operators are $P_0 \equiv |0\rangle\langle 0|$ and $P_1 \equiv |1\rangle\langle 1|$ (or more precisely, $P_0 = |0\rangle\langle 0| \otimes I \otimes I$ and $P_1 = |1\rangle\langle 1| \otimes I \otimes I$, where I is the identity operator for a qubit). Since the GHZ state is symmetric under permutations of the three qubits, it is enough to consider the measurement of a qubit, say, A. Then the state of the system after the measurement is $|000\rangle$ or $|111\rangle$, both of which are completely separable. If we associate the measurement with cutting of the corresponding ring, it means that the three rings are separated by cutting only one of the three rings. It is very nature of the famous Borromean rings (Fig. 3).

However, there are many possible measurements for a qubit. For example, let us consider a measurement corresponding to the spin measurement along x-direction.

The measurement operators are $P_+ \equiv |+\rangle\langle+|$ and $P_- \equiv |-\rangle\langle-|$, where

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{15}$$

$$|-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{16}$$

The GHZ state can be rewritten as

$$|\mathrm{GHZ}\rangle = \frac{1}{2}\{|+00\rangle + |+11\rangle + |-00\rangle - |-11\rangle\}. \tag{17}$$

Therefore the state after the measurement is

$$|+00\rangle + |+11\rangle = |+\rangle \otimes (|00\rangle + |11\rangle) \tag{18}$$

or

$$|-00\rangle - |-11\rangle = |-\rangle \otimes (|00\rangle - |11\rangle), \tag{19}$$

both of which have entanglement between qubit B and qubit C. Thus the correspondence introduced by Aravind depends on the choice of measurement basis.

We use the partial trace instead of measurement as a counterpart of the cutting of a ring. Physically speaking, it means that we just ignore a qubit, and observe only the other two. If we "trace out" qubit A in the GHZ state, the density operator for the remaining system is

$$\rho^{BC} \equiv \mathrm{tr}_A \rho = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right). \tag{20}$$

This is a separable mixed state, because it is represented by an ensemble consisting of separable pure states $|00\rangle$ and $|11\rangle$. [1] Therefore this state corresponds to the Borromean rings. Thus we can establish a connection between qubits and rings in a basis-independent way.

The next example is the so-called W state

$$|\mathrm{W}\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{21}$$

This state also has the permutation symmetry. If we trace out qubit A, the reduced density matrix is

$$\rho^{BC} = \frac{1}{3}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right), \tag{22}$$

[1] Aravind described this state as an "entangled" state, and associated it with two rings which cannot be pulled apart. It is normal, however, to classify it as an unentangled state.

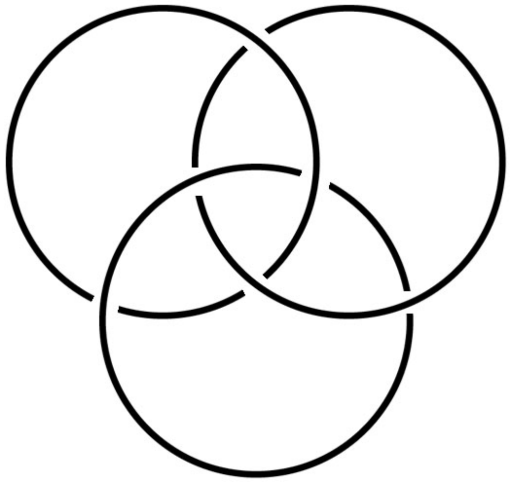

Figure 4: Three rings corresponding to the W state. If any ring is cut, the other two rings are still linked.

which is not separable. Actually we can show the non-separability by explicit evaluation of the concurrence as follows. The density matrix ρ^{BC} and its conjugate $\tilde{\rho}^{BC}$ can be written in the matrix form as

$$\rho^{BC} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\rho}^{BC} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}. \tag{23}$$

Hence

$$\tilde{\rho}^{BC}\rho^{BC} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2/9 & 2/9 & 0 \\ 0 & 2/9 & 2/9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Therefore $\lambda_1 = 2/3$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $C(\rho^{BC}) = 2/3 > 0$, which means that, in contrast to the Borromean rings, each pair of rings cannot be separated even after the third ring is cut. This case is modeled by the "three-Hopf rings" in Fig. 4.

There is yet another type of 3-qubit entanglement. Let us consider the following state

$$\begin{aligned} |\psi\rangle &= a|000\rangle + b|+1+\rangle & (25) \\ &= a|000\rangle + \frac{b}{2}(|010\rangle + |011\rangle + |110\rangle + |111\rangle). & (26) \end{aligned}$$

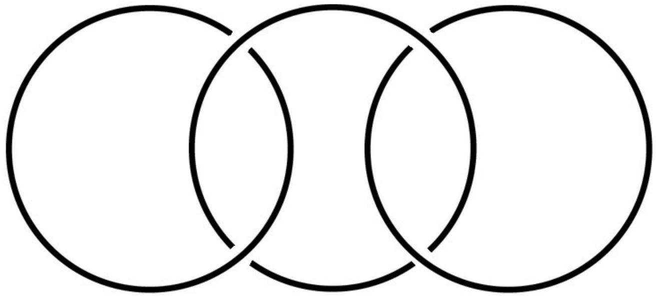

Figure 5: Linear-chain configuration of three rings. If the central ring is cut, the other two rings are unlinked. But if either edge ring is cut, the other two remain linked.

with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$. This state does not have the full permutation symmetry, but is symmetric with respect to qubit A and qubit C. If we trace out qubit B, the remaining state is

$$\rho^{AC} = a^2|00\rangle\langle 00| \; + \; b^2|++\rangle\langle ++|, \tag{27}$$

which is separable. If we trace out qubit A or C, however, the remaining state is still entangled. For example, if we trace out qubit C, the reduced density operator ρ^{AB} and its conjugate $\tilde{\rho}^{AB}$ can be written in the matrix form as

$$\rho_{AB} = \begin{pmatrix} a^2 & 0 & ab/2 & ab/2 \\ 0 & 0 & 0 & 0 \\ ab/2 & 0 & b^2/2 & b^2/2 \\ ab/2 & 0 & b^2/2 & b^2/2 \end{pmatrix}, \quad \tilde{\rho}_{AB} = \begin{pmatrix} b^2/2 & -b^2/2 & 0 & ab/2 \\ -b^2/2 & b^2/2 & 0 & -ab/2 \\ 0 & 0 & 0 & 0 \\ ab/2 & -ab/2 & 0 & a^2 \end{pmatrix} \tag{28}$$

Hence

$$\tilde{\rho}_{AB}\rho_{AB} = \frac{ab}{4}\begin{pmatrix} 3ab & 0 & 2b^2 & 2b^2 \\ -3ab & 0 & -2b^2 & -2b^2 \\ 0 & 0 & 0 & 0 \\ 4a^2 & 0 & 3ab & 3ab \end{pmatrix}. \tag{29}$$

Then we obtain $\lambda_1 = \frac{\sqrt{2}+1}{2}|ab|$, $\lambda_2 = \frac{\sqrt{2}-1}{2}|ab|$, $\lambda_3 = \lambda_4 = 0$, and hence $C(\rho^{AB}) = C(\rho^{BC}) = |ab|$, which is positive if $ab \neq 0$. This result means that the state (26) is modeled by the rings of linear-chain type in Fig. 5.

We have shown some examples of the correspondence between quantum entanglement and topological entanglement for some 3-qubit cases. It would be interesting to construct such correspondences for systems with more qubits. For example, Aravind argued a correspondence between the generalized GHZ state

$$\frac{1}{\sqrt{2}}(|00\ldots 0\rangle + |11\ldots 1\rangle) \tag{30}$$

and the generalized Borromean rings. Although our method does not uniquely determine a topological link corresponding to a quantum state, visualization of quantum states by topological objects could be a useful tool for the study of quantum entanglement.

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